Illustration of the dynamics of one and two dimensional kinematic wave equation

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The following are a couple of examples from a book I am currently writing about Evolutionary Distributions.

We are dealing with the quazilinear PDE

\[ a(x, y, u) \partial_x u + b(x, y, u) \partial_y u = c(x, y, u). \]  

(1)

**Example 0.1** Consider the so-called kinematic wave equation

\[ \partial_t u + u \partial_x u = 0. \]  

(2)

Solving for the characteristic ODE

\[ \frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = u, \quad \frac{du}{d\tau} = 0 \]  

(3)

with initial conditions \( t(0) = 0, \ x(0) = s \) and \( u(s, 0) = f(s) \), we obtain the parametric solution

\[ t(s, \tau) = \tau, \quad x(s, \tau) = \tau f(s) + s, \quad u(s, \tau) = f(s). \]

Because \( s = x - tf(s) = x - tu \), the general solution is

\[ u(x, t) = f(x - tu). \]  

(4)

From (3), \( u \) is (not necessarily the same) constant on each characteristic and \( dx/dt = u \). The characteristics themselves are straight lines with varying slopes \( u \) and we may be in for some surprises.

Let us use the kinematic wave equation (2) with the initial parametric data \( f(s) = s^2 \). Then the projections are

\[ x = s + s^2 t \]
Figure 1: The parametric solution surface of (1) with $f(s) = \Gamma = s^2$ and $-1 \leq s \leq 1$. Arrows indicate characteristic and their projections for $s = -1.0$, $0.5$, $0.0$, $0.5$. $\Gamma_0$ is the projection of the initial curve $\Gamma$. The vertical broken line pierces the solution surface at the point where the characteristics projections for $s = -1$ and $s = 0$ intersect.
and the implicit solution is \( u = (x - tu)^2 \). Figure 1 illustrates what happens. The solution surface folds directly above each intersection point of pairs of characteristic projections. For example, at the intersection of the projections for \( s = 0 \) and \( s = -1 \), the surface has two corresponding values, 0 and 1. In other words, at \( t = 1 \), the solution is in two places, which is unacceptable for dynamical physical systems. To better visualize what the situation, observe the characteristic projections (Figure 2). For \( x < 0 \), the slopes, \( dt/dx (= 1/u) \) become steeper as \( x \) increases. For \( x > 0 \) the slopes become shallower. On the solution surface, each intersection represents different values of \( u \), which contradicts what we know about physical reality. The fact that \( u \) can have multiple values along characteristics only provides an alternative definition of characteristics. We shall get back to this point later. All of this behavior is an inevitable consequence of the dependence of the characteristic projections on \( u \), which cannot be the case for semilinear equations.

Example 0.2 We have seen such a phenomenon in Example 0.1 (see also Figure 1). Figure 3 is based on Example 0.1. Here we look at two cross sections, at \( \tau = 0^+ \) and at \( \tau = 4 \). Note the two distinct values of \( u \) at the latter case at say, \( x = 2 \). Figure 4 animates the parametric solution of the kinematic wave equation.

Example 0.3 We are looking for the region in the \( x, t \)-plane along which the solution of

\[
\partial_t u + u \partial_x u = 0 \quad , \quad t > 0, \\
u = \sin (x) , \quad 0 \leq x \leq \pi \quad , \quad t = 0
\]
is unique. The parametric form of the solution is
\[ t = \tau, \quad x = s + \tau \sin (s), \quad u = \sin (s). \]
The implicit form of the solution is then
\[ u = \sin (x - t \sin (s)) = \sin (x - tu). \]
Figure 5 illustrates what happens at \( t \) progresses. On the right, observe cross cuts at \( t = 0, 1, \ldots, 4 \) and the ever progressing folding. At \( x = \pi, |\partial_x u| \) becomes unbounded and \( u \) ceases to be a single-valued function of its arguments.

**Example 0.4** For \( n = 3 \), we have \( x := [x_1, x_2, x_3] \) and the quazilinear equation is
\[ \sum_{i=1}^{3} a_i (x, u) \partial_{x_i} u = c (x, u). \]
With \( s := [s_1, s_2] \), the characteristics and their projections are obtained from the solution of
\[ \frac{d x_i (s, \tau)}{d \tau} = a_i (x, u), \quad x_i (s, 0) = x_{i0} (s), \quad i = 1, 2, 3, \]
\[ \frac{d u (s, \tau)}{d \tau} = c (x, u), \quad u (s, 0) = u_0 (s). \]
In particular, consider
\[ a_1 = 1, \quad a_2 = -\eta_1 u, \quad a_3 = -\eta_2 u, \quad c = u - u^2, \quad 0 \leq t \leq 2, \quad -1 \leq x_1, x_2 \leq 1. \]
Figure 4:
With \( x_1 \) identified as time, this is a model in which mutation rates on two adaptive traits depend on the density, \( u \) and \( u \) grows according to the logistic. Then
\[
\dot{x}_1 = 1, \quad \dot{x}_2 = -\eta_1 u, \quad \dot{x}_3 = -\eta_2 u, \quad \dot{u} = u - u^2
\]
where dots are derivatives with respect to \( \tau \). We assign the following initial data
\[
x_0 = [0, s_1, s_2], \quad u_0 = f(s).
\]
Now
\[
J(s, 0) = \begin{vmatrix}
1 & -\eta_1 u & -\eta_2 u \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix} = 1 \neq 0
\]
and we know that a local solution exists. The parametric solution (in \( s, \tau \)) is
\[
\begin{align*}
x_1 &= \tau, \quad \tau \in [0, 2], \\
x_i &= s_i - \eta_i \log [1 + (e^\tau - 1)f(s)], \quad s_i \in [-1, 1], \quad i = 1, 2, \\
u &= \frac{e^\tau f(s)}{1 + (e^\tau - 1)f(s)}.
\end{align*}
\]
Now suppose that at \( x_1 = \tau = 0 \), a distribution of mutants invade in the center of \((x_2, x_3)\) and that the initial distribution is
\[
f(s) = \exp \left[ - \left( \frac{s_1}{\sigma_1} \right)^2 - \left( \frac{s_2}{\sigma_2} \right)^2 \right].
\]
The dynamics are shown as a sequence at \( \tau = 0, 0.75, 1.25 \) and 2 in Figure 6. Note the bending and slicing. We shall return to this example when we
Figure 6:
talk about ED for multiple adaptive traits. For now, Figure 7 illustrates the dynamics.