Photon Propelled Space Vehicles*

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Summary—The interplanetary trajectories of vehicles propelled by solar radiation pressure are analyzed, and are shown to be logarithmic spirals if thrust direction is constant with respect to the vehicle-sun line. The required thrust may be obtained with a solar sail.

Sail size as a function of trip time to Mars is determined for solar thrust, oriented tangent to the trajectory.

Solar propulsion is compared with chemical and electrical propulsion. It is shown that a solar-sail-powered space vehicle on a journey from earth to Mars operates with a payload and flight time penalty when compared with a ballistic vehicle. However, the work capacity per unit weight of a solar sail is calculated to be superior to an electrical engine, which in turn is vastly superior to a chemical engine when the work is compared on the basis of equal flight times.

Introduction

Forbes1 showed that a logarithmic spiral is the space trajectory which results when thrust is applied tangent to the flight path, and when the magnitude of thrust is proportional to the inverse square of distance to the sun. Since the required thrust is small, the use of a solar sail is suggested. In this paper it is shown that the application of an inverse square thrust tangent to the flight path results in a special case of a more general class of logarithmic spiral trajectories.

Although some questions have been raised concerning the practical feasibility of solar sail vehicles for space travel, comparison with other continuous-thrust engines indicates that solar sailing must be given serious consideration in the light of present-day technology.

Notation

\[ r = \text{distance from sun to vehicle,} \]
\[ \theta = \text{longitude angle measured from the point where logarithmic spiral trajectory intersects the orbit of earth,} \]
\[ \phi = \text{solar sail tilt parameter } = (\tan \phi)^{-1} = F_r/F_s, \]
\[ A = \text{area of sail,} \]
\[ F_r = \text{force per unit mass along radius vector,} \]
\[ F_s = \text{force per unit mass perpendicular to radius vector,} \]
\[ \beta = \text{angle between tangent to spiral and radius vector,} \]
\[ q = \text{constant in logarithmic spiral curve, } r = e^{q\theta}, \]
\[ \mu = \text{gravitational constant of the sun,} \]
\[ S = \text{solar constant,} \]
\[ m = \text{total mass of vehicle,} \]
\[ t = \text{time,} \]
\[ v = \text{velocity.} \]

Dots denote a differentiation with respect to time; subscripts 0 denote initial conditions; and subscripts f denote final conditions.

Equations of Motion

If the applied thrust is zero, and forces are resolved along a radius vector to the sun and perpendicular to the radius vector (circumferential), then the equations of motion in a central force field are:

\[ \ddot{r} - r \dot{\theta}^2 + \frac{\mu}{r^2} = 0, \]  
\[ \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = 0. \]

When forces per unit mass \( F_r \) and \( F_s \) are applied in the radial and circumferential directions respectively, then (1) and (2) become:

\[ \ddot{r} - r \dot{\theta}^2 + \frac{\mu}{r^2} = F_r, \]
\[ \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = F_s. \]

Assume that the trajectory is a logarithmic spiral of the form

\[ r = e^{q\theta}, \]  
where \( 1/q = \tan \beta = \text{a constant.} \) The geometry of the situation is described in Fig. 1.

Fig. 1—Geometry of logarithmic spiral trajectory.

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* Original manuscript received by the IRE, December 11, 1959
Eqs. (3) and (4) may then be solved for $r$ and $\theta$ as functions of time. Time differentiation of (5) provides:

$$\dot{r} = qr\dot{\theta},$$

(6)

$$\ddot{r} = q^2\dot{\theta} + q\dot{r}\dot{\theta} = q(\ddot{r} + r\dot{\theta}).$$

(7)

After performance of the indicated differentiation in (4), $F_\theta$ is shown to be

$$F_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = (\ddot{r} + r\dot{\theta}) + r\dot{\theta}.$$  

(8)

Substitution of (8) into (7) yields

$$\ddot{r} = q(F_\theta - \ddot{r}).$$

(9)

Now, $\dot{r}$ in (9) may be set equal to the $\dot{r}$ in (3) so that:

$$F_r + r\dot{\theta}^2 - \frac{\mu}{r^2} = qF_\theta - q\dot{\theta}.$$  

(10)

Now let the sail apply forces per unit mass $F_r$ and $F_\theta$ in a constant ratio $\rho$, such that

$$F_\theta = 1 = \frac{1}{\rho} \tan \phi,$$

(11)

where $\phi$ is the angle between the sail normal and the radius vector to the sun. $\phi$ is also the angle of incidence of sunlight on the sail, as shown in Fig. 1. By combining (6), (10), and (11), we obtain:

$$F_\theta(\rho - q) + r\dot{\theta}^2(1 + q^2) = \frac{\mu}{r^2}.$$  

(12)

Since solar pressure falls off as $1/r^2$, let $F_\theta = K_1/r^2$. Then (12) may be solved for $\dot{\theta}$, which yields

$$\dot{\theta} = \frac{\mu + K_1(q - \rho)}{r^2(1 + q^2)} = \frac{\mu + K_1(q - \rho)}{1 + q^2} e^{-2q\phi};$$

(13)

and, since

$$2\ddot{\theta} = \left[ \frac{\mu + K_1(q - \rho)}{1 + q^2} \right] (-3q)e^{-3q\phi},$$

(14)

then

$$\dot{\theta} = -3/2q\dot{\theta}^2.$$  

(15)

From (8):

$$F_\theta = 2q\dot{r}\dot{\theta} - 3/2r^2q\dot{\theta}^2 = \frac{q\dot{r}\dot{\theta}^2}{2} = \frac{K_1}{r^2}$$

(16)

or

$$K_1 = \frac{q}{2r^2} \left[ \frac{\mu + K_1(q - \rho)}{1 + q^2} \right],$$

(17)

and

$$K_1 = \frac{q\mu}{q^2 + q\rho + 2}.  $$

(18)

Thus,

$$F_\theta = \frac{q\mu}{r^2(q^2 + q\rho + 2)},$$

(19)

and

$$F_r = \frac{\rho q\mu}{r^2(q^2 + q\rho + 2)}.$$  

(20)

Eq. (13) may be integrated to yield

$$e^{3/2q\phi} = 3/2q \sqrt{\frac{\mu + K_1(q - \rho)}{1 + q^2}} l + e^{3/2q\phi},$$

(21)

where $\phi = \theta_0$ when $t=0$.

Thus (5) and (21) yield

$$r^{3/2} = \frac{3q}{2} \sqrt{\frac{\mu + K_1(q - \rho)}{1 + q^2}} l + r e^{3/2q\phi}.$$  

(22)

**Tangential Thrust**

In the case where the force is applied tangent to the rocket path, then $q$ equals $\rho$, $\beta = \phi$, and (19), (20), and (21), (22) respectively reduce to:

$$F_r = \frac{\mu \rho^2}{2\rho^2(\rho^2 + 1)},$$

(23)

$$F_\theta = \frac{\mu \rho}{2\rho^2(\rho^2 + 1)},$$

(24)

$$e^{3/2q\phi} = 3/2q \sqrt{\frac{\mu}{\rho^2 + 1}} l + e^{3/2q\phi},$$

(25)

$$r^{3/2} = 3/2q \sqrt{\frac{\mu}{\rho^2 + 1}} l + r e^{3/2q\phi}.$$  

(26)

**Sail Area as a Function of Trip Time**

The case of thrusting tangent to the trajectory results in larger sail areas than necessary, but serves as a simple illustration of the method of determining the sail size. Later, we will show sail areas obtained for optimized tilt angles.

Fig. 2 represents the geometrical and force relationships of the sail with respect to the incident and reflected radiation from the sun.

A sail of area $A$ is oriented so that light incident upon it makes an angle $\phi$ with the normal to the sail. The projected area of the sail, perpendicular to the incident radiation, is $A \cos \phi$. If $S$ is the solar constant at Earth, and $S \cos \phi$ the radiation pressure normal to the sail, then the component of force in the radial direction must equal $mF_r$. Thus,

$$SA \cos^3 \phi = \frac{\mu \rho^2 m}{2\rho^2 \rho^2 + 1},$$

(27)
If the journey starts at Earth and ends at Mars, then
\[
 r_f = 1.524 \text{ AU}, \quad r_0 = 1.000 \text{ AU}
\]
and
\[
 \mu = 0.0002959 \text{ (AU)}^2/\text{day}^2,
\]
and
\[
 \frac{(p^2 + 1)^{1/2}}{p} = \frac{1}{34.16 \text{ days}}. \tag{33}
\]

The resultant area of sail required per unit mass is given in Table I in cm²/gram and in ft²/lb for various transfer times to Mars. The transfer angle, \( \theta \), measured from the start of the logarithmic spiral trajectory, is also listed.

**TABLE I**

<table>
<thead>
<tr>
<th>Transfer time (days)</th>
<th>( \frac{p}{(p^2 + 1)^{1/2}} )</th>
<th>Sail area per unit mass or earth weight</th>
<th>Transfer angle, ( \theta ) in radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0.427</td>
<td>77X10⁸ cm²/gm</td>
<td>0.9</td>
</tr>
<tr>
<td>100</td>
<td>0.342</td>
<td>97X10⁸ cm²/gm</td>
<td>1.2</td>
</tr>
<tr>
<td>150</td>
<td>0.228</td>
<td>140X10⁸ cm²/gm</td>
<td>1.8</td>
</tr>
<tr>
<td>200</td>
<td>0.171</td>
<td>190X10⁸ cm²/gm</td>
<td>2.4</td>
</tr>
<tr>
<td>250</td>
<td>0.137</td>
<td>240X10⁸ cm²/gm</td>
<td>3.1</td>
</tr>
<tr>
<td>300</td>
<td>0.114</td>
<td>300X10⁸ cm²/gm</td>
<td>3.7</td>
</tr>
</tbody>
</table>

The increase in sail area as journey time increases is explained in the term \( \cos^3 \phi \) in (27). Although the required solar force decreases in magnitude as \( p \) decreases (i.e., as the spiral angle \( \phi \) increases), the corresponding reduction in effective sail area causes a net increase in required sail size. However, the energy necessary to inject the space vehicle into the spiral orbit is greater for shorter transfer times. Hence, optimization on a complete vehicle systems basis is indicated.

**VEHICLE ACCELERATION**

The total acceleration per unit mass (\( F_T \)) provided by the sail for the cases of tangential thrust may be determined from:
\[
 F_T = F_r^2 + F_0^2 \tag{34}
\]
and
\[
 F_T = \frac{\mu}{2r^2 (p^2 + 1)^{1/2}} = \frac{296}{\text{sec}^2} \frac{p}{\sqrt{\nu^2 + 1}}. \tag{35}
\]

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This acceleration decreases as the vehicle travels farther from the sun. Table II lists the $g$'s acceleration experienced by the sail in the vicinity of Earth and in the vicinity of Mars, as a function of transfer time.

### Table II

<table>
<thead>
<tr>
<th>Transfer Time (days)</th>
<th>$g$'s at Earth</th>
<th>$g$'s at Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>$13 \times 10^{-6}$</td>
<td>$5.6 \times 10^{-6}$</td>
</tr>
<tr>
<td>100</td>
<td>$10 \times 10^{-6}$</td>
<td>$4.5 \times 10^{-6}$</td>
</tr>
<tr>
<td>150</td>
<td>$6.9 \times 10^{-6}$</td>
<td>$3.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>200</td>
<td>$5.2 \times 10^{-6}$</td>
<td>$2.2 \times 10^{-5}$</td>
</tr>
<tr>
<td>250</td>
<td>$4.1 \times 10^{-6}$</td>
<td>$1.8 \times 10^{-5}$</td>
</tr>
<tr>
<td>300</td>
<td>$3.4 \times 10^{-6}$</td>
<td>$1.5 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

### The Application of Non-Tangential Thrust

From the foregoing equations, a sail tilt parameter, $\rho$, and hence the sail tilt angle, $\phi$, may be computed, thus minimizing the sail area for a given trajectory (i.e., for a given value of $g$). In general, these minima occur for tilt angles in the neighborhood of 30° to 40° with the optimum tilt angle increasing as $\beta$ approaches 90°. Fig. 3 shows the sail area per unit weight plotted against the tilt angle $\phi$ for trajectories having $\beta$ angles of 80° and 85°. It is immediately evident that sails sized for tangential thrust applications are excessive when compared to minimum sail areas, although the transfer times are slightly less. Fig. 3 also shows that the transfer time has a strong dependence upon the angle $\beta$ and a weak dependence upon the angle $\phi$. Thus, it is better to consider a trade-off between sail size and trajectory path than between sail size and sail tilt angle.

Fig. 4 shows that some control over flight time may be achieved by changing the tilt of the sail. By fixing a sail design point slightly above the minimum, the transit time may be increased or decreased by five to ten days with a small change in sail tilt. Of course, if the sail is originally set at minimum flight time for the given area and trajectory, then the transit time can only be increased by a sail tilt angle change in either direction.

### Solar Sail Propulsion Compared to Chemical Propulsion

Because the photon-powered vehicle has a very low acceleration capability (typically $10^{-4}$ to $10^{-6}$ $g$'s), it cannot leave the Earth under its own power. Some additional energy must be added to inject the vehicle into a logarithmic spiral trajectory. The speed of a vehicle in a logarithmic spiral trajectory is nearly the speed of a vehicle in a circular orbit at the same distance from the sun, but with a change in direction. After the vehicle has escaped from the gravitational field of the Earth, an additional increment of velocity, $\Delta V$ (see Fig. 5), must be applied to turn the velocity vector into the logarithmic spiral velocity requirement, where

\[
\Delta V^2 = V_e^2 \left[ \left( 1 + \frac{2q^2 + 2}{q^2 + q\rho + 2} \right) - 2 \sin \beta \left( \frac{2q^2 + 2}{q^2 + q\rho + 2} \right)^{1/2} \right].
\]

If $\Delta V$ is applied after escape from the Earth, the total velocity budget needed to enter a logarithmic spiral is 6.97 mps (Earth escape velocity) plus $\Delta V$. After escape
the vehicle already has earth orbital velocity, and $\Delta V$ adjusts the direction. Fortunately, the burnout velocity, $V_b$, need not be the sum of 6.97 mps plus $\Delta V$, if fuel is burned near the surface of the Earth. The minimum required burnout velocity ($V_b$) may be calculated by the following equation:

$$V_b^2 = 6.97^2 + \Delta V^2.$$  

(37)

Then

$$\Delta V_a = V_b - 6.97,$$  

(38)

where $\Delta V_a$ is the velocity increment needed, in addition to escape velocity, at burnout. Both $\Delta V$ and $V_b$ are listed in Table III.

### Table III

**Comparison of Solar Sail Vehicle with a Ballistic Vehicle for Flights to Mars**

<table>
<thead>
<tr>
<th>Injection angle $\beta$</th>
<th>$\Delta V$ (mps)</th>
<th>$V_b$ (mps)</th>
<th>Injection payload (pounds)</th>
<th>Sail weight (pounds)</th>
<th>Transfer Time (days)</th>
<th>Solar sail vehicle</th>
<th>Ballistic vehicle</th>
</tr>
</thead>
<tbody>
<tr>
<td>75°</td>
<td>4.8</td>
<td>8.5</td>
<td>580</td>
<td>110</td>
<td>143</td>
<td>94</td>
<td>125</td>
</tr>
<tr>
<td>80°</td>
<td>3.2</td>
<td>7.7</td>
<td>860</td>
<td>110</td>
<td>208</td>
<td>125</td>
<td>404</td>
</tr>
<tr>
<td>85°</td>
<td>1.6</td>
<td>7.2</td>
<td>1100</td>
<td>76</td>
<td>404</td>
<td>404</td>
<td>not possible</td>
</tr>
</tbody>
</table>

The same $\Delta V_a$ may be used to inject a ballistic vehicle of the same weight into an interplanetary (elliptical) trajectory. As an example, assume that a 1200-pound vehicle is boosted to 6.97 mps near the surface of the Earth. Let $\Delta V_n$ be supplied by a chemical rocket which is part of the 1200 pounds. After burnout, the rocket is dropped and the remainder is designated as injection payload (IPL). For the photon-powered vehicle, the injection payload includes the weight of the sail.

A comparison of the times of flight for a photon-powered vehicle and a ballistic vehicle, injected into their respective heliocentric orbits with the same initial conditions, is also shown in Table III. The times of flight are longer for the solar-powered vehicle, and the useful payload is less since the sail weight is part of the injection payload.

Assuming a sail density of $2 \times 10^{-4}$ lb/ft², the weight of the sail is listed in Table III. While these weights are not excessive, it can be seen that the controllable sail-area may limit the size of solar sail vehicles.

### Solar Sail Work Compared to Electrical Engine Work

Assume a hypothetical electric engine which has a power output capability of 0.1 kw per second per pound of engine weight, and an efficiency of 70 per cent. After a determination of the optimum specific impulse for a particular mission is made, the mass ratio, power, and finally, the engine weight may be computed.

With the above engine set as a standard, a comparison between a solar sail and electrically powered vehicles is shown in Table IV. Here the sail-powered vehicle shows a payload weight advantage. It thus appears that in some instances the solar sail-powered vehicle is worthy of more consideration.

### Table IV

**Comparison of an Electric Powered Vehicle with a Solar Sail Vehicle**

<table>
<thead>
<tr>
<th>$T_e$ (days)</th>
<th>Optimized specific impulse (seconds)</th>
<th>Injection payload (pounds)</th>
<th>Propellant for electric powered vehicle (pounds)</th>
<th>Solar sail weight (pounds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>6500</td>
<td>542</td>
<td>249</td>
<td>97</td>
</tr>
<tr>
<td>200</td>
<td>7500</td>
<td>822</td>
<td>300</td>
<td>100</td>
</tr>
</tbody>
</table>

### Acknowledgment

The authors wish to thank Miss B. Cain, H. May, and Miss O. Helming for their able assistance in the generation and preparation of this paper.